

Counting the number of solutions of equations
in groups by recurrences

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1 The basic theorem

Let $G = (G, *, e)$ be a finite group with support $G = \{g_1, g_2, \dots, g_n\}$, operation $*$ and identity element $g_1 = e$. The aim of this paper is to find recurrences for the number $N(T, k, a)$ of solutions of the equation:

$$x_1 * x_2 * \dots * x_k = a$$

where $a \in G$ and the variables x_i are limited to belonging to a given subset T of G . Let θ be the left regular representation of G extended to the group algebra ZG . If $T \subset G$ we pose $\gamma(T) = \sum_{g \in T} g \in ZG$.

We begin with the basic result:

Theorem 1.1 *Given $T \subset G$, let $A = \theta(\gamma(T)) \in \text{Mat}(n, Z)$. Then:*

1. $N(T, k, g_j) = A_{1,j}^k$.
2. *The sequence $N(T, k, g_j)$, $k \in \mathbb{N}$, is linearly recurrent with characteristic polynomial $f(x)$, where $f(x)$ is any polynomial s.t. $f(A) = 0$.*

Proof

1. Let $T = \{g_{i_1}, g_{i_2}, \dots, g_{i_m}\}$.

$$(\gamma(T))^k = (g_{i_1} + g_{i_2} + \dots + g_{i_m})^k = \sum_{j=1}^n N(T, k, g_j) g_j$$

in ZG .

Applying θ on both sides:

$$A^k = \sum_{j=1}^n N(T, k, g_j) \theta(g_j).$$

The first row of $\theta(g_j)$ is $(0, \dots, 1, \dots, 0)$ with 1 in the j^{th} place and 0 elsewhere, and the result follows. \square

2. By Theorem 1.6 in [3] the sequence A_{ij}^k (for fixed indices i, j) is linearly recurrent with any polynomial $f(x)$ s.t. $f(A) = 0$ and initial values $A_{ij}^0, A_{ij}^1, \dots, A_{ij}^{m-1}$ (if $\deg(f(x)) = m$). \square

Example 1.2 Let $G = S_n$ (the symmetric group of degree n), $T = \{n\text{-cycles}\}$, $a \in T$. By Corollary 4.2 of [5]

$$N(T, k, a) = n!^{-1} (n-1)!^k \sum_{h=0}^{n-1} (-1)^{h(k-1)} \binom{n-1}{h}^{1-k} \quad (1)$$

We know from Theorem 1.1 that this sequence is recurrent. We now find a characteristic polynomial. If n is odd, collecting some terms, we can rewrite (1) as:

$$N(T, k, a) = \sum_{h=0}^{\frac{n-1}{2}} C_h [(-1)^h h! (n-h-1)!]^{k-1} \quad (2)$$

where the coefficients C_h are rational numbers. From equation (2) and Theorem C.1. of [6] we see that the sequence $N(T, k, a)$ is recurrent with characteristic polynomial of degree $\frac{n+1}{2}$:

$$f_{\text{odd}}(n) = \prod_{h=0}^{\frac{n-1}{2}} (x - (-1)^h h!(n-h-1)!)$$

For example if $n = 7$ $N(T, k, a)$ is linearly recurrent of fourth degree with characteristic polynomial $x^4 - 612x^3 - 80928x^2 + 2073600x + 149299200$ and initial values $\{1, 180, 153072, 106173504\}$.

Let us suppose now that n is even. Of course, in this case, when k is even $N(T, k, a) = 0$. We consider the subsequence formed by the terms with k odd, $k = 2s + 1$. From equation (1) we obtain:

$$N(T, 2s + 1, a) = \sum_{h=0}^{n-1} D_h [((-1)^h h!(n-h-1)!)]^2]^s$$

which can be rewritten as

$$N(T, 2s + 1, a) = \sum_{h=0}^{\frac{n}{2}-1} D_h ([h!(n-h-1)!]^2)^s.$$

Then the subsequence $N(T, 2s + 1, a)$, $s = 0, 1, \dots$, is recurrent with characteristic polynomial

$$f_{\text{even}}(n) = \prod_{h=0}^{\frac{n}{2}-1} (x - (h!(n-h-1)!)^2)$$

of degree $\frac{n}{2}$. For example if $n = 6$ $N(T, 2s + 1, a)$ is recurrent of third degree with characteristic polynomial $x^3 - 15120x^2 + 10450944x - 1194393600$ and initial values $\{1, 5040, 69237504\}$.

2 Smaller degree of recurrence

As we have seen the sequence $N(T, k, a)$ is always linearly recurrent with degree at most $n = |G|$, for any subset T in which we confine the variables x_1, x_2, \dots, x_k .

Sometimes we can find recurrences of lower degree.

Definition 2.1 *A partition $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$ of G is said to be closed if $\forall h, k \in \{1, \dots, m\}$ the set-product $T_h * T_k$ is a disjoint union of elements of \mathcal{T} .*

We can write

$$\gamma(T_h) * \gamma(T_k) = \sum \lambda_{hk}^s \gamma(T_s)$$

in the algebra ZG , where λ_{hk}^s is the number of solutions of the equation $x * y = g$ where $x \in T_h, y \in T_k, g \in T_s$. This number does not depend on g itself but only on the fact that $g \in T_s$. Then $\lambda_{hh}^s = N(T_h, 2, g)$, with $g \in T_s$.

Of course

$$\underbrace{\gamma(T_h) * \gamma(T_h) * \dots * \gamma(T_h)}_{k \text{ times}} = \sum N(T_h, k, g_s) T_s \text{ where } g_s \in T_s.$$

We abbreviate $N(T_h, k, g_s)$ to $N(h, k, s)$.

Now let $A_h = \theta(\gamma(T_h))$, $h = 1, \dots, m$. Then the set $\mathcal{A} = \{A_h : h = 1, \dots, m\}$ satisfies:

$$\sum_{h=1}^m A_h = J \text{ where } J \text{ is the all one matrix.} \quad (3)$$

There exist natural numbers λ_{hk}^s s.t.

$$A_h^k = \sum_{s=1}^m \lambda_{hk}^s A_s. \quad (4)$$

The numbers λ_{hk}^s are those we are searching for, i.e.

$$A_h^k = \sum_{s=1}^m N(h, k, s) A_s. \quad (5)$$

If we compute A_h^k , the k -th power of A_h , the number $N(h, k, s)$ appears in the places of the first row of A_h^k where A_s has ones.

Let us define the set of matrices \mathcal{B} , $\mathcal{B} = \{B_h : h = 1, \dots, m\}$ where $(B_h)_{ij} = \lambda_{hi}^j$. By the following theorem we obtain recurrences of degree lower than $|G|$ when T is an element of a closed partition.

Theorem 2.2 *Let $T_h \subset G$ be an element of a closed partition \mathcal{T} . Then the sequence $N(T_h, k, g)$, $g \in G$, satisfies a recurrence of degree at most $m = |\mathcal{T}|$ with characteristic polynomial any polynomial $f(x)$ s.t. $f(B_h) = 0$, where the matrix B_h is defined by: $(B_h)_{ij} = \lambda_{hi}^j$.*

Proof Again by Theorem 1.6 of [3] it is enough to prove that $N(T_h, n+1, g) = (B_h^n)_{ht}$ for every $h = 1, \dots, m$ and $n \geq 1$, with $g \in T_t$. We prove it by induction.

For $n = 1$, $N(h, 2, t) = \lambda_{hh}^t = (B_h)_{ht}$.

Let us suppose $N(h, n, t) = (B_h^{n-1})_{ht}$.

Then:

$$(A_h)^n = \sum_t N(h, n, t) A_t = \sum_t (B_h^{n-1})_{ht} A_t$$

and

$$\begin{aligned} (A_h)^{n+1} &= \sum_t (B_h^{n-1})_{ht} A_h A_t = \sum_{t,s} (B_h^{n-1})_{ht} \lambda_{ht}^s A_s = \\ &= \sum_{t,s} (B_h^{n-1})_{ht} (B_h)_{ts} A_s = \sum_s (B_h^n)_{hs} A_s. \end{aligned}$$

It follows that $(B_h^n)_{hs} = N(h, n+1, s)$ by equation (5) and the independence of the A_s . \square

Corollary 2.3 *Let G and H be respectively a finite group and an automorphism group of G . Let $\mathcal{O} = \{O_1, O_2, \dots, O_m\}$ be the set of orbits and let $N(h, k, t)$ be the number of solutions of $x_1 * x_2 * \dots * x_k = g$, with $x_i \in O_h$ and $g \in O_t$. Then $N(h, k, t)$ is linearly recurrent with characteristic polynomial of degree at most m .*

Proof It follows from Theorem 2.2 and the fact that \mathcal{O} is a closed partition. \square

Remark 2.4 1. In the case of Corollary 2.3 the matrices A_h form an association scheme (see [1]), where $A'_h = A_v$ and A_v is the matrix corresponding to the orbit $O_v = O_h^{-1}$.

2. The characteristic polynomial can be computed as the minimum polynomial of the matrix B_h .

3. The set of conjugacy classes is a well-known example, with $H = \text{Inn}(G)$. Examples 1.2 and ?? fall in this case, where conjugacy classes are those of n -cycles and transposition. Let us observe that from Theorem 1.1 we could only suppose a recurrence of degree $n! = |S_n|$. Instead from Theorem 2.2 and Corollary 2.3 we know that the recurrence degree for equations in S_n with variables constrained in conjugacy classes is at most equal to the number of partitions of n .

3 Cyclic groups and random walks on the circle

Let Z_n be the additive cyclic group $Z_n = \{0, 1, \dots, n-1\}$, and $Z_n^* = \text{Aut}(Z_n)$.

If $H \leq Z_n^*$ acts on Z_n we get m orbits:

$$O_0 = O(0), O_1 = O(1) = H, \dots, O_i = O(g_i), 0 \leq i \leq m-1,$$

with a set of representatives $\mathcal{R} = \{g_0 = 0, g_1 = 1, g_2, \dots, g_{m-1}\}$. We know that $\mathcal{T} = \{O(g_i), 0 \leq i \leq m-1\}$ is a closed partition.

Let now consider the special case $H = \{\pm 1\}$.

If n is odd we have $\frac{n+1}{2}$ orbits with $\mathcal{R}_{\text{odd}} = \{0, 1, \dots, \frac{n-1}{2}\}$ and if n is even we have $\frac{n+2}{2}$ orbits with $\mathcal{R}_{\text{even}} = \{0, 1, \dots, \frac{n}{2}\}$.

Let z be the $n \times n$ circulant matrix with first row $[0, 0, \dots, 0, 1]$, that is the permutation matrix corresponding to the n -cycle $(1, 2, \dots, n)$.

The adjacency matrices of the well known "polygon scheme" determined by the action of H are:

1. if r is odd

$$A_0 = I_n, A_k = z^k + z^{-k} \text{ for } 1 \leq k \leq \frac{n+1}{2}$$

2. if r is even

$$A_0 = I_n, A_{\frac{n}{2}} = z^{\frac{n}{2}}, A_k = z^k + z^{-k} \text{ for } 1 \leq k \leq \frac{n+2}{2}.$$

We divide the circle in n equal parts labeled $0, 1, \dots, n-1$.

Let $P(k, a)$ be the probability that we get the vertex a starting from 0 and flipping k times a coin to decide if to move one step clockwise or counterclockwise.

Of course

$$P(k, a) = \frac{N(O(1), k, a)}{2^k}$$

Theorem 3.1 *Let $g(x) = x^m + b_1x^{m-1} + \dots + b_m$ be the characteristic polynomial of B_1 .*

The sequence $P(0, a), P(1, a), \dots, P(k, a), \dots$ is recurrent with polynomial:

$$f(x) = x^m + \sum_{h=1}^m \frac{b_h}{2^h} x^{m-h}.$$

Proof From proof of Theorem 2.2 we know that we find $P(k, a)$ in the first row of $(\frac{1}{2}B_1)^k$. The result follows because if $g(x)$ is the characteristic polynomial of B_1 then $f(x)$ is the characteristic polynomial of $\frac{1}{2}B_1$. \square

Example 3.2 Let $n = 7$. The matrix $\frac{1}{2}A_1$ is the double stochastic transition matrix of the Markov chain associated with this random walk (see [4] p. 82).

$$A_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$C = \frac{1}{2}B_1$ is the stochastic matrix

$$C = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

We find $P(k, 0)$, that is the probability that we come back to the origin 0 after k steps, in the place (1, 1) of C^k .

From Theorem 3.1 the sequence $P(k, 0)$, $k \in N$ is recurrent with polynomial $x^4 - \frac{1}{2}x^3 - x^2 + \frac{3}{8}x + \frac{1}{8}$ and initial values $\{1, 0, \frac{1}{2}, 0\}$.

This recurrence sequence is convergent to $\frac{1}{7}$ (in general the first row of C^k converges to $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, i.e. $\forall a \lim_{k \rightarrow \infty} P(k, a) = \frac{1}{n}$).

The polygon scheme is a particular polynomial scheme. Then the matrix B_1 is tridiagonal and has the form

$$B_1 = \begin{pmatrix} * & 1 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 2 & 1 & \dots & 1 & * \end{pmatrix} \quad (6)$$

for n odd, and

$$B_1 = \begin{pmatrix} * & 1 & \dots & 1 & 2 \\ 0 & 0 & \dots & 0 & 0 \\ 2 & 1 & \dots & 1 & * \end{pmatrix} \quad (7)$$

for n even (see [1] for the notation).

Let $B_1^{(n)}$ be the tridiagonal matrix of the polygon scheme with n vertices, and $g_n(x)$ be its minimum polynomial.

$$g_n = \prod_{h=0}^{\lfloor \frac{n}{2} \rfloor} (x - 2 \cos \frac{2\pi h}{n}) \quad (8)$$

We now see that g_n can be easily computed by recurrence.

Theorem 3.3 *The sequence $g_n(x)$ is recurrent with polynomial*

$$y^4 - xy^2 + 1 \tag{9}$$

and initial values $\{g_0(x), g_1(x), g_2(x), g_3(x)\} = \{0, x - 2, x^2 - 4, x^2 - x - 2\}$.

Proof

$$B_1^{(n)} = \left\{ \begin{array}{cccccc} * & c_1 & c_2 & \dots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \dots & a_{d-1} & a_d \\ k & b_1 & b_2 & \dots & b_{d-1} & * \end{array} \right\} \tag{10}$$

where

$$c_1 = c_2 = \dots = c_{d-1} = 1, a_1 = a_2 = \dots = a_{d-1} = 0,$$

$$k = 2, b_1 = b_2 = \dots = b_{d-1} = 1,$$

and for n odd:

$$c_d = 1, a_d = 1, n = 2d + 1$$

for n even:

$$c_d = 2, a_d = 0, n = 2d.$$

Let us consider the sequence $F_0(x) = 1, F_1(x) = x + 1, F_i(x) = (x - k + b_{i-1} + c_i)F_{i-1}(x) - b_{i-1}c_{i-1}F_{i-2}(x)$.

Then ([1] pag 202):

$$(x - 2)F_d(x) = g_n(x).$$

If n is odd we have

$$F_i = xF_{i-1}(x) - F_{i-2}(x) \quad (11)$$

$\forall i, 2 \leq i \leq d$, which implies immediately

$$g_n(x) = xg_{n-2}(x) - g_{n-4}(x), \quad (12)$$

and (9) is proved.

If n is even eq. (11) holds true $\forall i, 2 \leq i < d$, but $F_d = (x+1)F_{d-1} - F_{d-2} = xF_{d-1} + F_{d-1} - F_{d-2}$.

Then

$$(x - 2)F_d = (x - 2)(xF_{d-1} - F_{d-2}) + (x - 2)F_{d-1}$$

that is

$$g_n(x) = g_{n+1}(x) + g_{n-1}(x) \quad (13)$$

with $n = 2d$. Hence

$$xg_{n-2} - g_{n-4} = x(g_{n-1} + g_{n-3}) - (g_{n-3} + g_{n-5}) = g_{n+1} + g_{n-1} = g_n \quad (14)$$

by (13) and (12). \square

Of course the sequence $g_k(x)$ has a geometrical meaning only if $k \geq 3$; we have extended it adding $g_0(x), g_1(x)$ and $g_2(x)$ by computing the recurrence backward.

Remark 3.4 Let $F_d^{\text{even}} = \frac{g_{2d}}{(x-2)}$ and $F_d^{\text{odd}} = \frac{g_{2d+1}}{(x-2)}$. Theorem (3.3) is equivalent to say that the sequence $F_0^{\text{even}}, F_1^{\text{even}}, \dots$ and $F_0^{\text{odd}}, F_1^{\text{odd}}, \dots$ are both recurrent with characteristic polynomial $y^2 - xy + 1$, with initial values respectively $\{1, x + 1\}, \{0, x + 2\}$.

Theorem 3.5 *Let C be the matrix*

$$\begin{pmatrix} x+1 & x+2 \\ -1 & -1 \end{pmatrix} \quad (15)$$

Then the first row of C^d is

$$[F_d^{\text{odd}}, F_d^{\text{even}}] \quad \forall d \geq 0.$$

Proof

The characteristic polynomial of C is $y^2 - xy + 1$ which is by theorem (3.3) the recurrence polynomial of both F_d^{odd} and F_d^{even} . Then the result follows from Remark 3.4 and Theorem 2.5 of [2], where the ring R is $Z[x]$. \square

Corollary 3.6 *The first row of $(x - 2)C^d$ is $[g_{2d+1}(x), g_{2d}] \quad \forall d \geq 0$.*

4 Dihedral group

Let D_n be the group of symmetries of a regular polygon $D_n = \{\rho^k, \tau\rho^k, k = 0, 1, \dots, n-1\}$, where n is the number of sides of the polygon, ρ is a rotation of $\frac{2\pi}{n}$ and τ is a reflection.

When n is odd, the regular representation θ is a direct sum of irreducible representations: $\theta = \psi_1 + \psi_2 + 2\phi_1 + 2\phi_2 + \dots + 2\phi_{\frac{n-1}{2}}$ where ψ_1 is the trivial representation, ψ_2 is the alternating representation and ϕ_l is the two dimensional representation such that

$$\phi_l(\rho^k) = \begin{pmatrix} \alpha^{lk} & 0 \\ 0 & \alpha^{-lk} \end{pmatrix} \quad \phi_l(\tau\rho^k) = \begin{pmatrix} 0 & \alpha^{-lk} \\ \alpha^{lk} & 0 \end{pmatrix}, \quad \alpha = \exp \frac{2\pi i}{n}$$

If n is even $\theta = \psi_1 + \psi_2 + \psi_3 + \psi_4 + 2\phi_1 + 2\phi_2 + \dots + 2\phi_{\frac{n-2}{2}}$ where $\psi_3(\rho^k) = \psi_4(\rho^k) = (-1)^k$ and $\psi_3(\tau\rho^k) = (-1)^k, \psi_4(\tau\rho^k) = (-1)^{k+1}$.

Let us now consider the case of two reflections, that generates D_n, τ and $\tau\rho$, i.e. suppose $T = \{\tau, \tau\rho\}$ and $a \in D_n$.

Theorem 4.1 1. *The sequence $N(T, k, a)$ is recurrent with polynomial*

$$p_n(x) = \frac{g_{2n}^2(x)}{x^2 - 4}. \quad (16)$$

2. *The sequence $p_n(x)$ for $n = 1, 2, \dots$ is recurrent with polynomial*

$$y^4 - y^3x^2 + (2x^2 - 2)y^2 - x^2y + 1 \quad (17)$$

and initial values $\{x^2 - 4, x^4 - 4x^2, -4 + 9x^2 - 6x^4 + x^6, -16x^2 + 20x^4 - 8x^6 + x^8, -4 + 25x^2 - 50x^4 + 35x^6 - 10x^8 + x^{10}\}$.

Proof

1. From the decomposition of θ , if n is even

$$p_n(x) = x^2(x-2)(x+2) \prod_{h=1}^{\frac{n-2}{2}} (x^2 - 4 \cos^2 \frac{2\pi h}{n})^2 \quad (18)$$

and if n is odd

$$p_n(x) = (x-2)(x+2) \prod_{h=1}^{\frac{n-1}{2}} (x^2 - 4 \cos^2 \frac{2\pi h}{n})^2. \quad (19)$$

Collecting appropriate terms and using equation (8) we find (16). \square

2. By remark 3.4 $p_n(x) = \frac{x-2}{x+2}(F_n^{\text{even}}(x))^2$. In the ring $Z(x) \frac{x-2}{x+2}$ is constant and the sequence $p_n(x)$ is recurrent with the same recurrence of $(F_n^{\text{even}}(x))^2$. By the same remark $F_n^{\text{even}}(x)$ is recurrent with poly-

nomial $y^2 - xy + 1$ whose companion matrix is $C = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$.

By Theorem 2.6 of [2] $(F_n^{\text{even}}(x))^2$ is recurrent with the characteristic polynomial of the Kronecker product $C \otimes C$, that is $y^4 - y^3x^2 + (2x^2 - 2)y^2 - x^2y + 1$. \square

For example if $n = 7$ the sequence $N(T, k, e)$ is recurrent with polynomial $-4 + 49x^2 - 196x^4 + 294x^6 - 210x^8 + 77x^{10} - 14x^{12} + x^{14}$ and initial values $\{0, 2, 0, 6, 0, 20, 0, 70, 0, 252, 0, 924, 0, 3434\}$.

We consider now the case of the basic rotation ρ and the reflection τ , i.e. $T = \{\rho, \tau\}$ and $a \in D_n$.

Theorem 4.2 1. *The sequence $N(T, k, a)$ is recurrent with polynomial*

$$p_n^{odd}(x) = \frac{g_n^2}{(x-2)}x^n \quad (20)$$

if n is odd and

$$p_n^{even}(x) = \frac{g_n^2}{(x-2)(x+2)}x^n \quad (21)$$

if n is even.

2. *The subsequences p_{2s+1}^{odd} and p_{2s}^{even} are recurrent with polynomial*

$$y^4 - y^3x^4 + (2x^6 - 2x^4)y^2 - x^8y + x^8 \quad (22)$$

and initial values respectively $\{x^2 - 2x, -2x^3 - 3x^4 + x^6, -2x^5 + 5x^6 - 5x^8 + x^{10}, -2x^7 - 7x^8 + 14x^{10} - 7x^{12} + x^{14}\}$ and $\{-4x^6 + x^8, -4x^6 + 9x^8 - 6x^{10} + x^{12}, -16x^{10} + 20x^{12} - 8x^{14} + x^{16}, -4x^{10} + 25x^{12} - 50x^{14} + 35x^{16} - 10x^{18} + x^{20}\}$.

Proof

1. From the decomposition of θ , we find

$$p_n^{\text{even}}(x) = x^2(x-2)(x+2) \prod_{h=1}^{\frac{n-2}{2}} x^2(x - 2 \cos \frac{2\pi h}{n})^2 \quad (23)$$

and

$$p_n^{\text{odd}}(x) = x(x-2) \prod_{h=1}^{\frac{n-1}{2}} x^2(x - 2 \cos \frac{2\pi h}{n})^2. \quad (24)$$

Equations (20) and (21) follows from (8). \square

2. In the ring $Z[x]$ $p_{2s+1}^{\text{odd}}(x)$ is equal to $x(x-2)$ multiplied by $(F_s^{\text{odd}}(x))^2 x^{2s}$.

Now $(F_s^{\text{odd}}(x))^2$ is recurrent by characteristic polynomial $y^4 - y^3x^2 + (2x^2-2)y^2 - x^2y + 1 = u(x)$ and x^{2s} by $y-x^2$. We use again Theorem 2.6 of [2]: the characteristic polynomial of x^2U , where U is the companion matrix of $u(x)$, is precisely $y^4 - y^3x^4 + (2x^6 - 2x^4)y^2 - x^8y + x^8$.

The same holds for $p_{2s}^{\text{even}}(x)$. \square

For example if $n = 7$ the sequence $N(T, k, e)$ is recurrent with polynomial $-2x^7 - 7x^8 + 14x^{10} - 7x^{12} + x^{14}$ and initial values

$$\{0, 1, 0, 3, 0, 10, 1, 35, 9, 126, 55, 462, 286, 1717\}.$$

If $n = 8$ the sequence $N(T, k, e)$ is recurrent with polynomial $-16x^{10} + 20x^{12} - 8x^{14} + x^{16}$ and initial values

$$\{0, 1, 0, 1, 0, 3, 0, 10, 0, 36, 0, 136, 0, 528, 0, 2080, 0, 8256\}.$$

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